1. (16 points) Let $G(V, E)$ be a complete graph on $n$ vertices. A tournament $T(V, E)$ is nothing but an orientation of the edges of $G(V, E)$. Specifically, for any two vertices $u$ and $v$ in $V$, exactly one of the directed edges $(u, v)$ and $(v, u)$ is present in $T(V, E)$. A Hamiltonian path in a tournament is a directed path passing through all the vertices. Show that there is a tournament on $n$ vertices that has at least $\frac{n!}{2^{n-1}}$ Hamiltonian paths.

2. (17 points) Input are $k$ sorted sequences $X_1, X_2, \ldots, X_k$ such that $\sum_{i=1}^{k} |X_i| = n$. The problem is to merge these $k$ sequences. Show how to solve this problem in $O(\log k \log \log n)$ time using $n$ CREW PRAM processors.

3. (17 points) Input are $k$ sets represented as (not necessarily sorted) sequences each of length $n$. Keys in these sequences are integers in the range $[1, n \log^{10} n]$. Show how to compute the intersection of these sets in $O(\log n \log k)$ time. You can use up to $\frac{nk}{\log n}$ Arbitrary CRCW PRAM processors.

Extra credit: Show that the above problem can be solved in time $\tilde{O} \left( \min \left\{ \log n \log k, \log n \log \log n + \frac{\log n}{\log \log n} \log k \right\} \right)$ using $\frac{nk}{\log n}$ Arbitrary CRCW PRAM processors.

4. (17 points) Input is an array $A[1 : n]$ of arbitrary real numbers. The problem is to find an element of $A$ whose rank in $A$ is in the interval $[\frac{3n}{8}, \frac{5n}{8}]$. Present a Monte Carlo algorithm to solve this problem that runs in $O(\log n)$ time. You can use up to $n^{1/4}$ CREW PRAM processors. Prove that the output of your algorithm will be correct with high probability.

5. (17 points) Let $A$ and $B$ be two $n \times n$ matrices of arbitrary real numbers. The tropical product of $A$ and $B$ is a matrix $C$ such that $C[i, j] = \min_{k=1}^{n} A[i, k] + B[k, j]$, for $1 \leq i, j \leq n$.

Present a Las Vegas algorithm that takes as input two $n \times n$ matrices and computes the tropical product of these in $O(1)$ time. You can use up to $n^3$ arbitrary CRCW PRAM processors.

6. (16 points) Input are a sequence $X$ of $n$ arbitrary real numbers (not necessarily in sorted order) and another real number $r$. The problem is to check if there are two elements $a$ and $b$ of $X$ whose sum is $r$. Present a $O(\log n)$-time algorithm for this problem. You can use up to $n$ arbitrary CRCW PRAM processors. Assume that the numbers in $X$ are distinct.