Name: ________________________________

CSE 6512 Randomization in Computing

Exam I; October 24, 2023

Note: You are supposed to give proofs to the time bounds of your algorithms. Read the questions carefully before attempting to solve them.

1. (16 points) Input is an array $A[1:n]$ of arbitrary real numbers. The array could only be of one of the following two types: 1) **Type I:** A has $\frac{n}{3}$ elements that are in the range $[1,2)$, another $\frac{n}{3}$ elements in the range $[2,3)$, and another $\frac{n}{3}$ elements in the range $[3,4]$; or 2) **Type II:** A has $\frac{n}{3}$ elements that are in the range $[2,3)$, another $\frac{n}{3}$ elements in the range $[3,4]$, and another $\frac{n}{3}$ elements in the range $(4,5]$. Present a Las Vegas algorithm that determines the type of the array in $\widetilde{O}(\log n)$ time.

2. (17 points) Input is a sequence X of n arbitrary real numbers and an integer i, $1 \leq i \leq n$. The problem is to output an element of X whose rank in X is in the interval $i \pm \widetilde{O}(n^{3/4})$. Present a Monte Carlo algorithm for this problem that takes $O(n^{2/3})$ time. Prove the correctness of your algorithm.

3. (17 points) Input are $n \times n$ matrices $A_1, A_2, \ldots, A_k, C$. The problem is to check if C is the inverse of $\Pi_{i=1}^{k} A_i$. Present a Monte Carlo algorithm to solve this problem in $O(n^2 k)$ time. Show that the output of your algorithm will be correct with a high probability.

4. (16 points) Input are polynomials $f_1(x), f_2(x), \ldots, f_k(x)$, the degree of $f_i(x)$ being d_i, $1 \leq i \leq k$, such that $\sum_{i=1}^{k} d_i = n$. Let $F(x) = \Pi_{i=1}^{k} f_i(x)$. Input are also the polynomials $g_1(x), g_2(x), \ldots, g_\ell(x)$, the degree of $g_i(x)$ being e_i, $1 \leq i \leq \ell$, such that $\sum_{i=1}^{\ell} e_i = n$. Let $G(x) = \Pi_{i=1}^{\ell} g_i(x)$. The problem is to check if $F(x) = G(x)$. Present a Monte Carlo algorithm to solve this problem that runs in $O(n)$ time. Show that the output of your algorithm will be correct with a high probability.

5. (17 points) We have shown in class that a random skip list with n elements has a height of $\widetilde{O}(\log n)$. Prove that the space needed to store a random skip list with n elements in $\widetilde{O}(n)$.

6. (17 points) Let $M = \{0, 1, \ldots, m-1\}$ and $N = \{0, 1, \ldots, n-1\}$, with $m \geq n$. A family H of hash functions from M into N is said to be strongly 2-universal if for all $x_1, x_2 \in M, x_1 \neq x_2$, and any $y_1, y_2 \in N$, and h chosen uniformly randomly from H, $Prob.[h(x_1) = y_1 \text{ and } h(x_2) = y_2] = \frac{1}{n^2}$. Consider the case where $m = n = p$, where p is a prime number. Show that the following hash family is strongly 2-universal: $H = \{h_{a,b}(x) = ((ax+b) \bmod p) \bmod n \ : \ a, b \in \mathcal{Z}_p\}$.