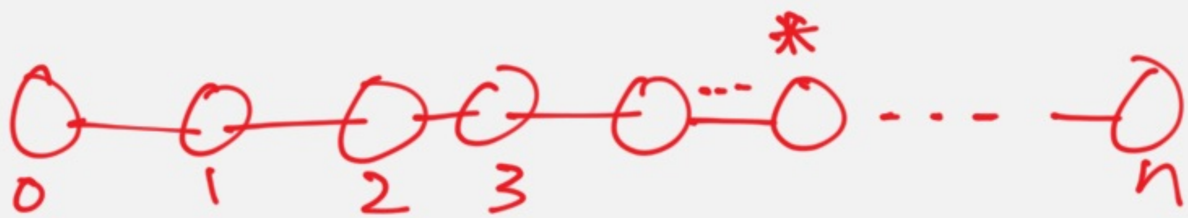
CSE 6512 RANDOMIZATION IN COMPUTING

LECTURE: 10-10-2023

Let F be a formula in 2SAT

Let $S_1, S_2, \ldots, S_n$ be a SATISFYING assignment.



# of CORRECT ASSIGNMENTS →

A MARKOV CHAIN M IS Characterized

With S → a set of States.

TRANSITION PROB. MATRIX P.

P_{ij} = if the current state of M is i, this is the prob. that the next state is j.

let X_t be the state of M at time step t.

M is MEMORYLESS.

I.E.,

$$\text{Prob}[X_t = i / X_{t-1} = j, X_{t-2} = j_{t-2}, X_{t-3} = j_{t-3}, \ldots, X_1 = j_1, X_0 = j_0]$$

$$= \text{Prob}[X_t = i / X_{t-1} = j] = P_{ji}$$

$$p_{ij}^{(t)} = \text{Prob}\left[X_t = j / X_0 = i\right]$$

$r_{ij}^{(t)}$ = Prob that $X_t = j$, given that $X_0 = i$ and state j has not been visited in steps $1, 2, \ldots, t-1$.

Let f_{ij} = Prob. of ever visiting node j given that $X_0 = i$.

$$f_{ij} = \sum_{t>0} r_{ij}^{(t)}$$

Let h_{ij} = Expected time to visit j given that $X_0 = i$.

$$h_{ij} = \sum_{t>0} t \; r_{ij}^{(t)}$$

If $f_{ij} < 1$ then $h_{ij} = \infty$.

IF $f_{ii} < 1$ then the node i is SAID to be TRANSIENT.

IF $f_{ii} = 1$ then i IS PERSISTENT.

IF $f_{ii} = 1$ and $h_{ii} = \infty$ then the node i IS NULL PERSISTENT.

IF $f_{ii} = 1$ and $h_{ii} < \infty$ then the node i is NON NULL PERSISTENT.

NOTE: We can represent a Markov Chain as a directed graph $G(V,E)$ $V = S$. $\langle i,j \rangle \in E$ IF $P_{ij} > 0$.

DEFN. S IS A STRONG COMPONENT IN A DIRECTED GRAPH $G(V,E)$ IF S is the maximal subgraph s.t. $\exists$ A DIRECTED PATH FROM EVERY NODE to every other node.

IF the graph of a MARKOV CHAIN CONSISTS OF A STRONG COMPONENT then EVERY NODE IS PERSISTENT. We say M IS IRREDUCIBLE.

IF M IS A FINITE MARKOV CHAIN, then EACH State IS EITHER TRANSIENT OR NON-NULL PERSISTENT.

The State PROB. VECTOR OF M at TIME Step t, $q^{(t)}$

$$q^{(t)} = \left(q_1^{(t)}, q_2^{(t)}, \ldots, q_n^{(t)} \right)$$

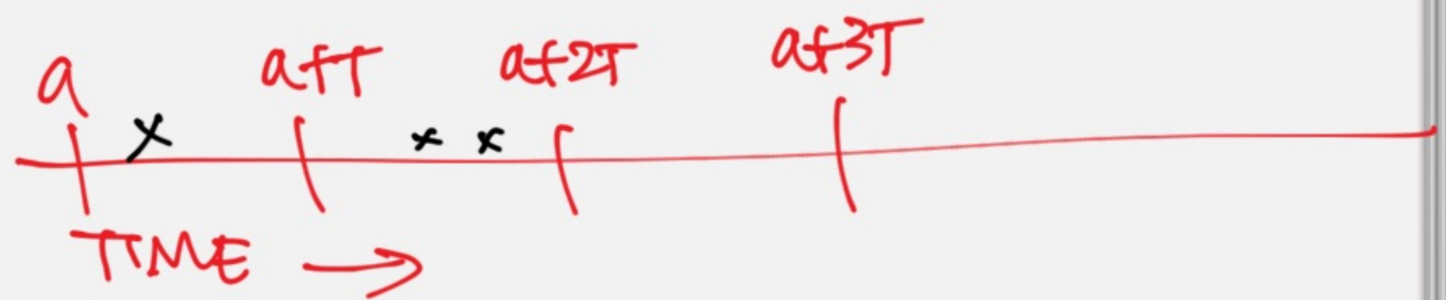
where $q_i^{(t)}$ = PROB. of the State of M is i at time step t.

$$q^{(t+1)} = q^{(t)} P$$

We call π as the STATIONARY STATE PROB. VECTOR IF:

$$\pi = \pi P.$$

Let i be any node of M.
We say that the period of i is T if T is the largest integer s.t. if $q_i^{(t)} > 0$ then $t \in \{a + qT : q \geq 0\}$, FOR SOME INTEGER a.



WE SAY A STATE IS APERIODIC IF
ITS PERIOD IS 1.

A MARKOV CHAIN IS APERIODIC IF
EVERY STATE IS APERIODIC.

A STATE IS ERGODIC IF IT IS APERIODIC and NON NULL PERSISTENT.

A MARKOV CHAIN IS ERGODIC IF EVERY STATE IS ERGODIC.

THEOREM: IF M IS FINITE, IRREDUCIBLE and APERIODIC then the following statements ARE TRUE.

1. EVERY STATE OF M IS ERGODIC
2. $\exists$ A UNIQUE STATIONARY PROB. VECTOR. $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$
3. $\forall i$: $f_{ii} = 1$; $h_{ii} = \frac{1}{\pi_i}$
4. $\pi_i = \mathop{Lt}\limits_{t \to \infty} \frac{N(i, t)}{t}$

WHERE $N(i, t) = \#$ of TIME Steps DURING WHICH M WAS IN State i IN t SUCCESSIVE TIME Steps.

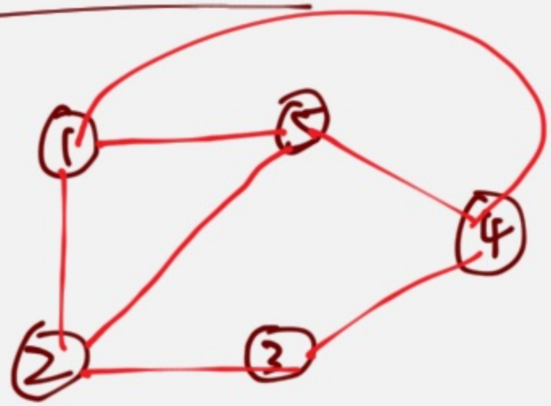
RANDOM WALKS ON GRAPHS:

Let G BE AN UNDIRECTED, CONNECTED & NON-BIPARTITE GRAPH.

We can define a Markov chain M FOR G(V,E) as follows:

$S = V.$ $\qquad p_{ij} = \frac{1}{d_i}$

Example:



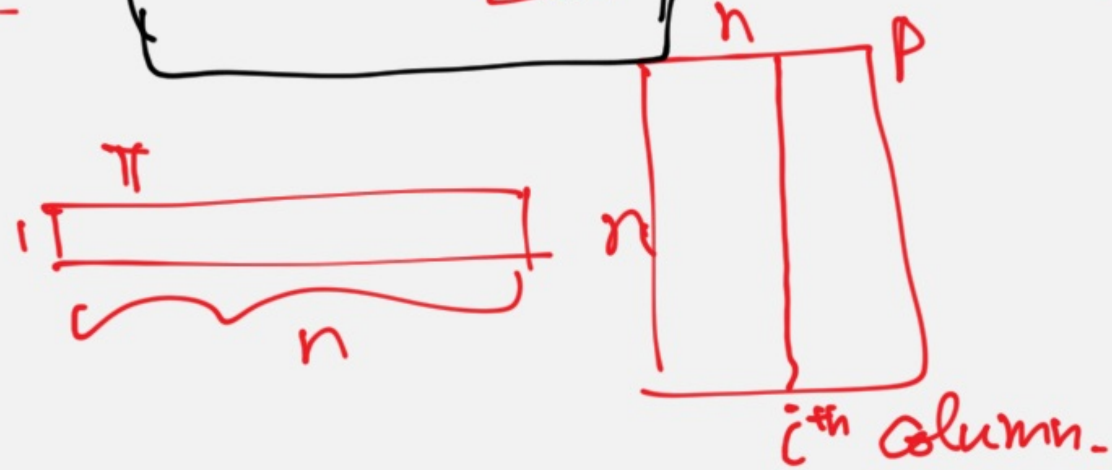
$$P \quad \begin{array}{c|ccccc} j \downarrow \backslash i \rightarrow & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 2 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 3 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 4 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 5 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \end{array}$$

PERIOD = GCD(CYCLE LENGTHS).

$\Rightarrow$ PERIOD = 1 $\forall$ nodes.

CLAIM: $\boxed{\pi_i = \frac{d_i}{2m}}$ where $m = |E|$.

PROOF:



$$(\pi P)_i = \pi_1 P_{1i} + \pi_2 P_{2i} + \cdots + \pi_n P_{ni}$$

$$= \sum_{(j,i)\in E} \pi_j P_{ji} = \sum_{(j,i)\in E} \frac{d_j}{2m} \cdot \frac{1}{d_j}$$

$$= \sum_{(j,i) \in E} \frac{1}{2m} = \frac{d_i}{2m} . \quad \square$$

$$h_{ii} = \frac{1}{\pi_i} = \frac{2m}{d_i}$$

Let G BE A CONNECTED, UNDIRECTED, and NON-BIPARTITE GRAPH. then:

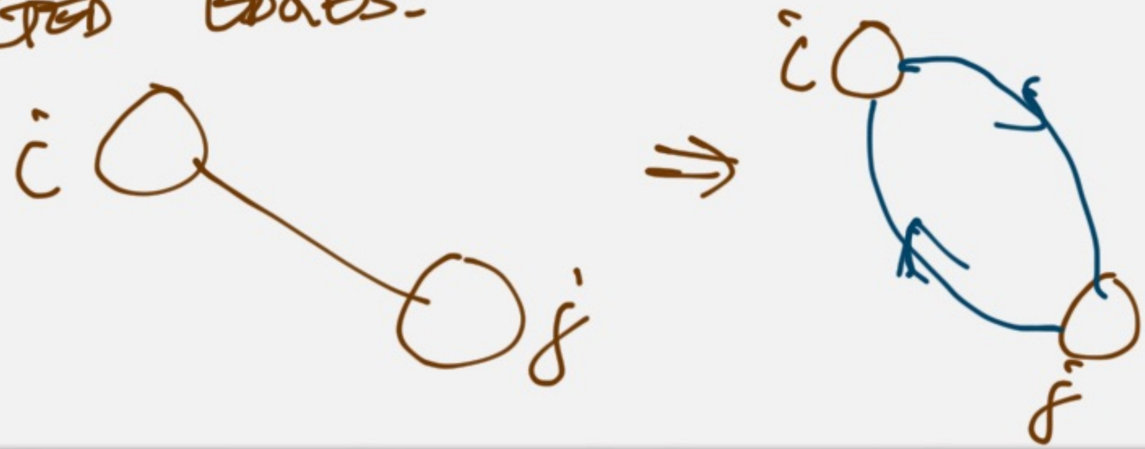
① What $h_{ij} + h_{ji}$ for any nodes i and j? COMMUTE TIME BETWN. i and j.

(2) Let $C_i(G)$ be the expected time to visit EACH node at least once given that the start state is i.

(3) COVER TIME $= \max_{i \in V} C_i(G)$.

Let (i,j) be any edge in the graph.
Then $h_{ij} + h_{ji} \leq 2m$, $m = |E|$.

PROOF: CONSTRUCT A MARKOV CHAIN M'. M' is CONSTRUCTED FROM G' where G' is CONSTRUCTED FROM G. G' is obtained FROM G as follows: REPLACE EVERY EDGE with 2 DIRECTED EDGES:



Nodes of M′ will be directed edges of the form: $\langle i, j \rangle$.

Define the TRANSITION PROB. MATRIX Q AS FOLLOWS:

$$Q_{\langle i,j\rangle \langle j,k\rangle} = \frac{1}{d_j}$$

$$Q_{\langle a,b\rangle \langle c,d\rangle} = \begin{cases} \frac{1}{d_b} & \text{IF } b=c \\ =0 & \text{OTHERWISE.} \end{cases}$$

Q IS DOUBLY STOCHASTIC.

$$\sum_{x,y \in G} Q_{\langle x,y\rangle\langle j,k\rangle} = \sum_{\langle i,j\rangle \in E} Q_{\langle i,j\rangle\langle j,k\rangle}$$

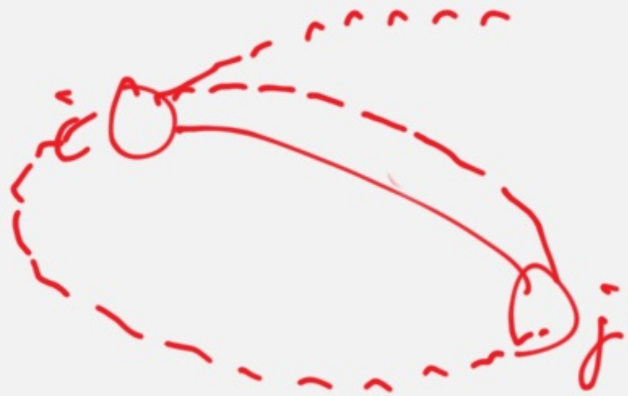
$$= \sum_{\langle i,j\rangle \in E} \frac{1}{d_j} = d_j \cdot \frac{1}{d_j}$$

$$= 1. \quad \square$$

$\Rightarrow$ π IS UNIFORM.

$$\Rightarrow \quad \pi = \left(\frac{1}{2m}, \frac{1}{2m}, \ldots, \frac{1}{2m} \right).$$

$\Rightarrow$ Expected time to visit $\langle i,j \rangle$ starting FROM $\langle i,j \rangle = \frac{1}{\pi_i} = 2m.$



$h_{ij} + h_{ji} \leq 2m.$