CSE 4502/5717 Big Data Analytics
Fall 2022; Homework 2 Solutions

1. Dijkstra's algorithm can be described as follows:

   Algorithm 1: Dijkstra(V, E, s)

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   Data: (V, E): a graph;
       s: a source node;
       let w(u, v) be the weight of edge (u, v);
   Result: array d where d_u is the length of the shortest path from s to u;
   begin
       for u in V do
           d_u := ∞;
       d_s := 0;
       Create a priority queue Q to store pairs of the form (node, distance);
       Insert the pair (s, 0) into Q;
       while Q not empty do
           (u, r) := ExtractMin(Q);
           for every child c of u do
               if d_c > d_u + w(u, c) then
                   d_c := d_u + w(u, c);
                   Insert(Q, (c, d_c)); // update distance if c present
   ```

   We assume that we can store the priority queue in memory ($O(|V|)$). The algorithm will read the neighbors of each node at most once. Therefore, the total number of I/Os is $\sum_{u \in E} \lceil \frac{deg_u}{B} \rceil = O\left(\frac{|E|}{B} + |V|\right)$.

2. We apply the LMM algorithm with $l = m = \sqrt{M}$. We assume known that we can merge $\sqrt{M}$ sequences of length M each in 3 passes through the data. The pseudocode of the algorithm is given below:

Algorithm 2: Sort(X, N)

Data:

X: array of elements;

$N = M^2$: number of elements in X;

Result: sorted array X;

begin

// First Pass;

Split the input into M runs of length M each;

Sort each run and unshuffle it into $m = \sqrt{M}$ sequences of length $\sqrt{M}$ each;

// Second Pass;

Merge groups of $l = \sqrt{M}$ unshuffled sequences (in memory);

// Third Pass;

Shuffle groups of $m = \sqrt{M}$ merged sequences of length M each;

At the same time clean up the dirty regions;

At this point we have $\sqrt{M}$ sorted runs of length $M\sqrt{M}$ each;

// Third Pass (can be done with the previous pass);

Unshuffle each run of length $M\sqrt{M}$ into $m = \sqrt{M}$ sequences of length M each ;

// Fourth, Fifth and Sixth Pass;

Merge groups of $l = \sqrt{M}$ unshuffled sequences of length M each;

// Seventh Pass;

Shuffle groups of $m = \sqrt{M}$ merged sequences of length $M\sqrt{M}$ each;

Clean up dirty regions;

For an arbitrary N, the general principle is to first merge $\sqrt{M}$ sequences of length M each, then merge $\sqrt{M}$ sequences of length $M\sqrt{M}$ each and so on. Let K stand for $\sqrt{M}$ and let $T(u, v)$ be the number of passes required to merge u sorted sequences of length v each. Then we have the familiar formulas:

$$
\begin{aligned}
T(K, M) &= 3 \\
T(K, K^i M) &= 2 + T(K, K^{i-1}M) = 2i + 3 \\
T(K^c, M) &= T(K, M) + T(K, KM) + T(K, K^2 M) + \ldots + T(K, K^{c-1}) \\
&= \sum_{i=0}^{c-1} (2i + 3) = c^2 + 2c
\end{aligned}
$$

However, as we saw in the previous pseudocode, when we compute $T(K^c, M)$ we can overlap the unshuffling at the beginning of a $T(K, K^i M)$ computation with the shuffling done at the end of the previous $T(K, K^{i-1}M)$ computation. Therefore, the last equation becomes:

$$T(K^c, M) = T(K, M) + \ldots + T(K, K^{c-1}) - (c-1) = c^2 + c + 1$$

Therefore the number of passes for M^2 and M^3 elements are:

$$\begin{aligned} T(M^2) &= T(M, M) = T(K^2, M) = 2^2 + 2 + 1 = 7 \\ T(M^3) &= T(M^2, M) = T(K^4, M) = 4^2 + 4 + 1 = 21 \; \square \end{aligned}$$

In general, for a given N, if $K^c = N/M$ it means that $c = 2\frac{\log N/M}{\log M}$ and the number of passes to sort N elements is:

$$T(N) = T(K^c, M) = 4\left(\frac{\log N/M}{\log M}\right)^2 + 2\frac{\log N/M}{\log M} + 1.$$

3. The input striping is good for accessing the rows of the matrix in a disk parallel manner. However, if we want to access the columns, this striping is not good. To multiply A and C we need the transpose of C. To get this, we first restripe the matrix C as follows. Let R_i be the ith row of C. We read R_i into core memory in $\frac{n}{DB}$ parallel I/Os. We then rewrite row R_i starting from disk i mod D (with one block per disk). This is done for every $1 \leq i \leq n$. After this restriping, we read one column at a time into the core memory and write it back to the disks one block per disk (starting from the first disk). Note that a column can be read in $\frac{n}{D}$ parallel I/O operations. Thus the matrix C can be transposed in $\frac{n^2}{D} < \frac{n^3}{DB}$ parallel I/O operations.

   We then use the following algorithm. Let $E = AC$.

   for $i := 1$ **to** n **do**

       Read row i of A into core memory. Let this row be called A_i.

       for $j := 1$ **to** n **do**

           Read column j of C into core memory. Let this column be C_j.

           $E_{ij} = \sum_{k=1}^{n} A_i[k] * C_j[k]$.

       Write row i of E into the disks, striping the data in a row-major order.

Each row or column of A or C can be read in $O\left(\frac{n}{DB}\right)$ parallel I/Os. Also, each row of E can be written in $O\left(\frac{n}{DB}\right)$ I/Os. Thus the total number of parallel I/Os is $O\left(\frac{n^3}{DB}\right)$.

4. Let the input strings be $S_1, S_2, \ldots, S_k$ with $\sum_{i=1}^{k} |S_i| = M$. Build a generalized suffix tree for these strings in $O(M)$ time. Let the suffixes be labelled with (i, j) where i refers to S_i and j refers to the jth suffix in S_i. Perform a depth first traversal in this tree.

   When we reach a leaf labelled $(i, 1)$ for some i, this leaf corresponds to the entire string S_i. This leaf might have more than one labels. Let these labels (in addition to $(i, 1)$) be $(i_1, l_1), (i_2, l_2), \ldots, (i_q, l_q)$. Clearly, all the strings $S_{i_1}, S_{i_2}, \ldots, S_{i_q}$ have S_i as a substring. Output all of these strings as those that contain S_i. Check if the edge to this leaf's parent is labeled with \$. If not, proceed with the traversal. If yes, let x be the parent of this leaf. Also, let $c_1, c_2, \ldots, c_r$ be the other children of x. Traverse through all the subtrees rooted at these children. All the leaves in these subtrees also correspond to strings that have S_i as a substring. Output these strings as well (as those that contain S_i) and proceed with the traversal.

   The entire algorithm can be implemented to run in time $O(M + k^2)$.

5. Let $S_1, S_2, \ldots, S_k$ be the given input strings. Let $|S_i| = n_i$, for $1 \le i \le k$. For any two strings S_i and S_j we can compute the longest common substring between them in $O(n_i + n_j)$ time, for $1 \le i, j \le k$. Use this algorithm to compute the longest common substring between every pair of strings. The total run time is $O(\sum_{i=1}^{k} \sum_{j=1}^{k} (n_i + n_j)) = O(kM)$.

6. Note that on a common CRCW PRAM we can compute the minimum or maximum of n integers (in the range $[1, n^{O(1)}]$) in $O(1)$ time using n processors.

   Let T be the text and P be the pattern with $|T| = m$ and $|P| = n$. We can use binary search on the suffix array. In any iteration of binary search, we have to compare the pattern P with a suffix T_i of the text. This comparison involves the identification of the smallest integer q such that $P[q] \ne T_i[q]$. This can be done in $O(1)$ time using the above algorithm. Thus the entire binary search takes $O(\log m)$ time.