CSE 6512 Lecture 22; November 10, 2011
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1 Integer sort

1.1 Coarse Sort

In Coarse Sort we have to sort n integers in the range $\left[1, \frac{n}{(\log n)^3}\right]$. Let $D = \frac{n}{(\log n)^3}$. Let the input be $X = k_1, k_2, ..., k_n$. Let $I(k) = \{i : k_i = k\}$, $1 \le k \le D$. The Coarse Sort algorithm works as follows.

1. Compute $N_1, N_2, ..., N_D$ such that $N_i \ge |I(i)|$ for $1 \le i \le D$ and $\sum_{i=1}^{D} N_i = O(n)$.

   a) For $1 \le i \le D \log n$ in parallel do: processor i picks randomly a key from X.

   b) Sort the sample picked in a) using Preparata's algorithm. This can be done in $O(\log n)$ time using $\frac{n}{\log n}$ processors). Let $I_s(k) = \{i : k_i$ is in the sample and $k_i = k\}$.

   c) For $1 \le i \le D$ in parallel do: Processor i computes $N_i = d\alpha \log^2 n \times \max\{|I_s(k)|, \log n\}$ where d is a constant. This is done in O(1) time.

2. Using the N_i's and the assignment algorithm, rearrange the keys. The group # of any key is its value.

1.2 Analysis

1. Case 1. If $|I(k)| < d\alpha \log^3 n$ then $N_k \ge I(k)$

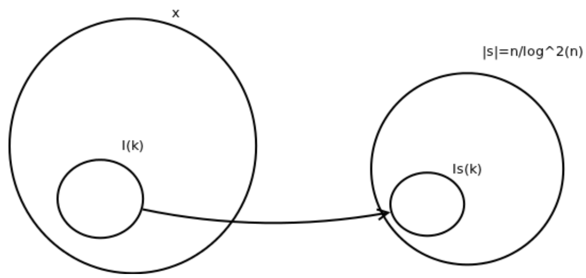
2. Case 2. $|I(k)| \ge d\alpha \log^3 n$:

   Note that $I_S(k)$ is binomial with parameters $(\frac{n}{\log^2 n}, \frac{|I(k)|}{n})$. Using Chernoff bounds:

   Prob[$|I_S(k)| < (1-\epsilon)\frac{|I(k)|}{\log^2 n}$] $\le \exp(\frac{-\epsilon^2 |I(k)|}{2\log^2 n})$

   Let $\epsilon = \frac{1}{2} \Rightarrow$ RHS $\le \exp(\frac{-|I(k)|}{8\log^2 n})$

   If $d \ge 8$, RHS $\le n^{-\alpha}$



$\Rightarrow N_k \geq I(k)$ with probability $\geq (1 - n^{-\alpha})$.

3. $\sum\limits_{k=1}^{D} N_k = \sum\limits_{k=1}^{D} d\alpha \log^2 n \times \max\{|I_s(k)|, \log n\}$

   $\leq \sum_{k=1}^{D} d\alpha \log^2 n\{|I_s(k)| + \log n\}$

   $\leq d\alpha \log^2 n \sum\limits_{k=1}^{D} |I_s(k)| + \sum\limits_{k=1}^{D} d\alpha \log^3 n$

   $= 2d\alpha n = O(n)$

   Note that $\sum\limits_{k=1}^{D} d\alpha \log^3 n = d\alpha n$,

   $\sum\limits_{k=1}^{D} |I_s(k)| = \dfrac{n}{\log^2 n}$,

   and $d\alpha \log^2 n \sum\limits_{k=1}^{D} |I_s(k)| = d\alpha n$.

2 Sub-logarithmic time algorithms

2.1 Solving the assignment problem in $O(\frac{\log n}{\log \log n})$ time

Input: $X = k_1, k_2, ..., k_n$

Group #'s: $g_1, g_2, ..., g_n$. Assume that the group number is an integer in the range $[1, q]$.

Upper bounds on the group sizes: $N_1, N_2, ..., N_q$

Output: A rearrangment of X based the groups #'s.

Theorem: We can solve the above problem in $O\left(\frac{\log n}{\log \log n}\right)$ time using $\frac{n}{\log n}(\log \log n)^2$ arbitrary CRCW PRAM processors.

Proof: Here is an algorithm: Let $P = \frac{n}{\log n}(\log \log n)^2$.

1. Using a prefix sums computation on $2N_1, 2N_2, ..., 2N_q$, identify the boundaries of the buckets. This takes $O(\frac{\log n}{\log \log n})$ time.

2. For $1 \leq i \leq n$ in parallel do:

   Make $\log \log n$ attempts to place the key k_i in its right bucket.
   This can be done using $\frac{n}{\log n}(\log \log n)^2$ processors in $O(\frac{\log n}{\log \log n})$ time.

3. Do a prefix computation to identify the number Z of elements that have not been placed yet.

   $\Rightarrow$ This takes $O(\frac{\log n}{\log \log n})$ time.

4. Assign $(\frac{P}{Z})$ processors to each such element. The processors associated with any such element attempt in parallel to place the element in its bucket. A total of $O(\frac{\log n}{\log \log n})$ time is allocated.

5. Even if there is a single unsuccesful element, start all over again (from step 2).

Analysis: in the next lecture.