Randomized algorithms

Notes of lecture 21

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Taken By

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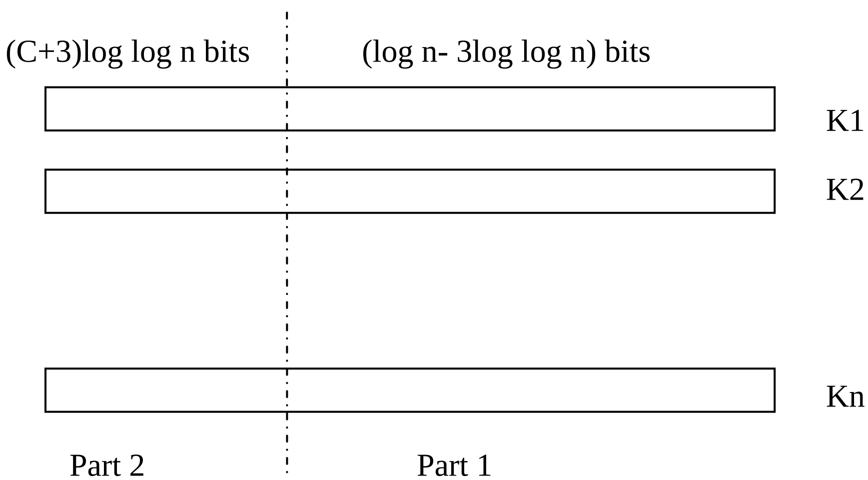
Theorem:

We can sort n integers in the range $[1, n(\log n)^C]$ in $\tilde{O}(\log n)$ time using n/log n Arbitrary CRCW PRAM processors, where c is any constant.

Proof: Here is an algorithm:

There are two phases:



Think of each integer as a binary string.

Phase 1 - Sort the keys with respect to their first part -> Coarse Sort

Phase 2 - Stable sort the keys with respect to their second parts -> Fine Sort

Fact: - If ∃ a stable sort algorithm that sorts n integers in the range [1,R] in time T using P processors, we can use the same algorithm to sort n integers in the range $[1, R^C]$ in O(T) time using P processors, for any constant C.

Fine Sort:

Lemma – we can sort n integers in the range [1, log n] in O(log n) time using n/log n CREW PRAM processors.

Proof: Let $\boxed{K_1, K_2, \ldots, K_{\log n}}$, $\boxed{K_{\log n+1}, \ldots, K_{2\log n},}$ … $\boxed{\ldots\ldots\ldots, K_n}$ be the input.

Number of processors, P = n/log n

Assign log n keys per processor

Step 1

For 1≤i≤P in parallel do

Processor i performs a bucket sort of its keys and computes N_{ij} = # of keys with a value j 1≤j≤log n.

N_{11} N_{12} N_{13} ……………………………………… $N_{1\ logn}$

N_{21} N_{22} N_{23} ……………………………………… $N_{2\ logn}$

. . .

. . .

N_{P1} N_{P2} N_{P3} ……………………………………… $N_{P\ logn}$

This step takes O(log n) time.

Step 2

All the P = n / log n processors do a prefix sums computation on N_{11}, N_{21}, N_{31}, N_{P1}, N_{12}, N_{22}, …., N_{P2}, ……., $N_{1\ log\ n}$, $N_{2\ log\ n}$, ………, $N_{P\ log\ n}$

This step takes O(log n) time.

Step 3

For 1≤i≤P in parallel do

processor i writes its keys with value j,

starting from memory cell

$N_{11}+N_{21}+$…………………$N_{P1}+ N_{12} + N_{22}+$…………$N_{P2}+$……….$+N_{1(j-1)} +N_{2(j-1)}+$ ……+

$N_{P(j-1)}+N_{1j}+N_{2j}+\ldots+N_{(i-1)j}+1$.

The above is done for each value of j, 1≤j≤log n.

This step takes O(log n) time.

Total runtime for this algorithm = O(log n)

Note: This algorithm is stable

Corollary: We can stable sort n integers in the range $[1,(\log n)^C]$ in $O(\log n)$ time using $n/\log n$ CREW PRAM Processor, for any constant C. This takes care of the Fine Sort problem.

An assignment problem

Input is a sequence

$X = k_1, k_2, \ldots, k_n.$

Each key k_i belongs to a group g_i

Think of g_i as an integer.

$1 \leq i \leq n$; $1 \leq g_i \leq Q$.

Let n_j be the # of keys that belong to group j, $1 \leq j \leq Q$.

Let N_i be such that

$N_i \geq n_i$ and $\sum N_i = O(n)$.

Given X and the N_i's the problem is to permutate the sequence X, such that all the keys in group1 appear first, followed by all the keys in group 2, …, followed by all the keys in group Q.

***LEMMA:** The above assignment problem can be solved in $\tilde{O}(\log n)$ time using $n/\log n$ Arbitrary CRCW PRAM processors.

Bucket1	Bucket 2	………………………………	Bucket Q
$2N_1$	$2N_2$		

(linear space)

Assume that the estimates Ni's are in common memory (as a part of the input)

N_1	N_2	………………….	N_Q

Assign log n keys per processor

Step 1

Perform a prefix computation on

$N_1, N_2, \ldots, N_Q$ and assign

$2N_i$ cells for group i,

$1 \leq i \leq Q$

Step 2

. For $1 \leq i \leq P = n/\log n$ in parallel do

Repeat

Processor i picks the next unassigned element and performs as many rounds as needed to place it.

Until all the log n elements are placed.

A Round – Let k be the element to be placed & let q be the group #.
Pick a random cell in bucket q
If this cell is occupied, wait for the next round; If not try to write k in this cell;
Read from this cell; if the cell has k, move on to next key; If not wait for the next round.

Step 3

Compress the buckets using a prefix sums computation.

Analysis

For any processor, probability of success in any round is $\geq \frac{1}{2}$

- ⇨ Expected # of keys successfully placed in one round $\geq \frac{1}{2}$
- ⇨ The # of keys successfully placed in $d\,\alpha\log n$ rounds is lower bounded by a binomial $B(d\alpha\log n, \frac{1}{2})$ using Chertoff bounds, this # is >= log n with probability $\geq (1-n^{-\alpha})$ for some proper constant d.

A possible value for d is 16.