Lecture 13

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MARKOV CHAINS

A Markov Chain is a discrete time stochastic process characterized by S, a set of states and P, a transition probability matrix.

S could either be finite or <u>countably</u> infinite:

$$S=\{s_1, s_2, s_3, \ldots\}$$

$P_{i\,j}$ - the probability that the next state is j, given that the current state is i.

Let X_t be the state of the Markov Chain at time step t.

Markov Chain is <u>memoryless</u>.

i.e., Probability $[X_t = j/X_{t-1}=i, X_{t-2}=i_1, \ldots, X_0 = i_{t-1}]$= Probability $[X_t = j/ X_{t-1}=i]= P_{i\,j}$

$P^{(t)}_{ij}$ is known as the t-step probability, where

$$P^{(t)}_{ij} = \text{Probability } [X_t = j/X_0 = i]$$

Definition:

Let $r^{(t)}_{ij}$ be the probability that the state of the Markov Chain at step t is j and X_0 = i and state j has not been visited in the steps 1, 2, … , t-1.

Let $f_{i\,j}$ be the probability that the state j will ever be visited if X_0 = i.

$$f_{i\,j} = \sum_{t>0} r^{(t)}_{ij}$$

Definition:

Let h_{ij} be the expected time to visit state j starting from state i at step 0.

$$h_{ij} = \sum_{t>0} t\ r^{(t)}_{ij}$$

<u>*Note:*</u> if $f_{ij} < 1$ then $h_{ij} = \infty$.

Definition:

If $f_{ii} < 1$ for any state i, then it is said to be TRANSIENT.

If $f_{ii} = 1$ for state i, then it is said to be PERSISTENT.

Definition:
If for any state i, f_{ii} =1 and $h_{ii} = \infty$, then the state is NULL-PERSISTENT.

Definition:
If for any state i, f_{ii} =1 and $h_{ii} < \infty$, then the state is NON-NULL PERSISTENT.

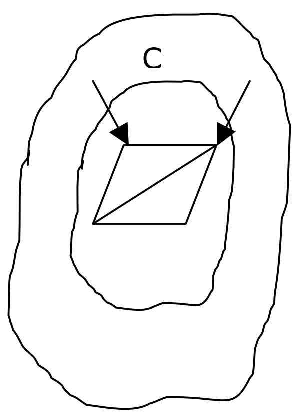
FACT:
In any finite Markov Chain, each state is either TRANSIENT or NON-NULL PERSISTENT.

FACT:
We can use a directed graph to represent a Markov Chain.
Nodes $\longrightarrow$ States

$\exists$ an edge from i to j in G(V, E) if P_{ij} is > 0.

Definition:
In a directed graph, a STRONG COMPONENT refers to a Maximal subgraph in which there is a directed path from every node to every other node.

Definition:
In a directed graph, a FINAL STRONG COMPONENT refers to a STRONG COMPONENT such that there is no edge going from a node in this component to a node outside this component.



FACT:
If M is a finite Markov Chain, and C is a STRONG COMPONENT, then $\exists$ a non-zero probability of starting from any node of C and reaching any other node in C in a finite number of steps.
If C is a FINAL STRONG COMPONENT, then this probability is 1.

FACT:

If a Markov Chain consists of a single STRONG COMPONENT, then all the states are PERSISTENT. We call this Markov Chain as IRREDUCIBLE.

Definition:

The state probability vector of a Markov Chain at step t is denoted as

$$q^{(t)} = (q_1^{(t)}, q_2^{(t)}, \dots , q_n^{(t)})$$

where, $q_i^{(t)}$ = probability that the state of the Markov Chain at step t is i, for $1 \leq i \leq n$.

FACT:

$$q^{(t+1)} = q^{(t)}P$$

where P is the probability transition matrix. This implies that, $q^{(t)} = q^{(0)} P^t$.

Definition:

The stationary probability distribution π satisfies:

$$\pi = \pi P$$

Definition:

The periodicity of a state i is the largest T such that if $q_i^{(t)} > 0$, then $t \in \{a+qt : q \geq 0\}$, for some integer a.

If T = 1 for any state, then the state is APERIODIC.

Definition:

A Markov Chain is APERIODIC if all of its states are APERIODIC.

Definition:

A state of a Markov Chain is ERGODIC if it is APERIODIC and NON-NULL PERSISTENT.

Definition:

A Markov Chain is ERGODIC if all of its states are ERGODIC.

THEOREM:

For any FINITE, APERIODIC and IRREDUCIBLE Markov Chain, the following are true:

1) All the states are ERGODIC.

2) ∃ a unique stationary probability distribution π, such that,

$\pi = (\pi_1, \pi_2 ..., \pi_n)$ with $\pi_i > 0$, $1 \leq i \leq n$.

3) For every state i, $f_{ii} = 1$, and $h_{ii} = (1/\pi_i)$, and

4) $\forall_i \; \pi_i = \text{Lim}_{t-\infty} \frac{N(i,t)}{t}$, where N(i,t) is the number of times the state of Markov Chain was i in t successive time steps.

RANDOM WALKS ON GRAPHS:

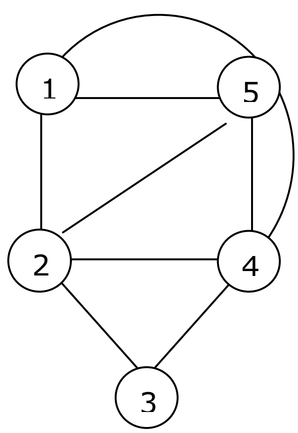
Let G(V,E) be any connected, undirected and non-bipartite graph. We can construct a Markov Chain M_G as follows:

State set = V.

If (u,v) ∈ E, then $P_{uv} = \frac{1}{d_u}$ where d_u is the degree of the node u.

Example:

G



$M_{G,}$ S = {1, 2, 3, 4, 5}

P

	1	2	3	4	5
1	0	1/3	0	1/3	1/3
2	1/4	0	1/4	1/4	1/4
3	0	1/2	0	1/2	0
4	1/4	1/4	1/4	0	1/4
5	1/3	1/3	0	1/3	0

Note: M_G is FINITE and IRREDUCIBLE.

Since G is a non bipartite, it has a cycle of odd length. Also, it has a cycle of length 2.

The PERIOD is the GCD of cycle lengths. Thus the period is 1. This implies that M_G is APERIODIC.

Applying the theorem, M_G has a unique stationary probability distribution.

LEMMA:

M_G has a unique π where $\pi_u = \frac{d_u}{2m}$, for u ∈ V.

PROOF:

$$\sum_{u \in V} \pi_u = \frac{\sum_{u \in V} d_u}{2m} = \frac{2m}{2m} = 1$$

Let n = |V|

$$[\pi P]_u = \pi_1 P_{1u} + \pi_2 P_{2u} + \ldots + \pi_n P_{nu}$$
$$= \sum_{(w,u) \in E} \pi_w P_{wu}$$
$$= \sum_{(w,u) \in E} \frac{d_w}{2m} \cdot \frac{1}{d_w}$$
$$= \frac{1}{2m} \sum_{(w,u) \in E} 1$$
$$= \frac{d_u}{2m}$$

FACT: $h_{uu} = \dfrac{1}{\pi_u} = \dfrac{2m}{d_u}$