CSE 6512 Lecture 11 Notes

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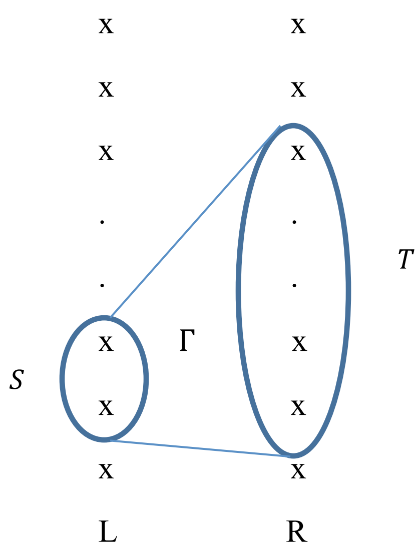
EXPANDER GRAPHS:

DEFN: A $(\mathrm{n}, \mathrm{d}, \alpha, \mathrm{C})$ OR-concentrator is a bipartite multigraph $G(L, R, E)$, with $|L| = |R| = n$. The degree of each node in L is d. For every subset S of L with $|S| \leq \alpha n$, $|\Gamma(S)|$ is $\geq C|S|$ where $\Gamma(S) =$ Neighbor set of S.

We want $d, \alpha, and\ C$ to be constants.

LEMMA: $\forall n \geq n_0, \exists\ a\ (n, 18, \frac{1}{3}, 2)$ OR-concentrator.

PROOF:



Let S be a specific subset of L.

Let T be a specific subset of R of size $Cs, where\ s = |S|$.

Generate the edges randomly (with replacement).

$$Prob[\,T\ contains\ all\ the\ neighbors\ of\ S] = (\frac{Cs}{n})^{ds}$$

$$\Longrightarrow Prob[\exists\ a\ T\ of\ size\ C|S|\ that\ contains\ all\ the\ neighbors\ of\ S] \leq \binom{n}{Cs}(\frac{Cs}{n})^{ds}$$

$\Longrightarrow Prob[\exists\ a\ subset\ S\ of\ size\ s\ all\ of\ whose\ neighbors\ are$

$$contained\ in\ some\ subset\ T\ of\ size\ Cs] \leq \binom{n}{s}\binom{n}{Cs}\left(\frac{Cs}{n}\right)^{ds}$$

We have $\binom{a}{b} \approx \left(\frac{ae}{b}\right)^{b}$.

$$RHS \approx \left(\frac{ne}{s}\right)^{s}\left(\frac{en}{Cs}\right)^{Cs}\left(\frac{Cs}{n}\right)^{ds} = P_s$$

$$P_s = n^{s+Cs-ds}s^{ds-Cs-s}e^{s+Cs}C^{ds-Cs}$$

$$= [\left(\frac{s}{n}\right)^{d-C-1}e^{1+C}C^{d-C}]^{s}$$

$$\leq [\left(\frac{1}{3}\right)^{d-C-1}e^{1+C}C^{d-C}]^{s} \leq \left[(3e)^{1+C}\left(\frac{C}{3}\right)^{d}\right]^{s}$$

$$< \left(\frac{1}{2}\right)^{s}$$

$\Longrightarrow Prob[\exists\ a\ subset\ S\ of\ size\ \leq \alpha n\ S.T.its\ neighbor\ set\ is\ of\ size\ \leq Cs]$

$$\leq \sum_{s=1}^{\alpha n} P_s < \sum_{s=1}^{\alpha n}\left(\frac{1}{2}\right)^{s} \leq \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

$\Longrightarrow This\ prob\ is\ < 1$

$\Longrightarrow The\ complement\ prob\ is\ > 0 \Longrightarrow Prob[G\ is\ an\ OR-concentrator] > 0$

$\Longrightarrow \exists\ a\ \left(n, 18, \frac{1}{3}, 2\right)\ OR-concentrator.$ **End of Proof.**

LOVASZ LOCAL LEMMA:

Let $E_1, E_2, \ldots, E_n$ be events with $Prob[E_i] \leq p, \forall i$ and each E_i is independent of all the other events except for d of them.

If $ep(d+1) \leq 1$

$then\ Prob[\bigcap_{i=1}^{n} \overline{E_i}] > 0.$

EXAMPLE:

Let $F = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ be a K-CNF Boolean Formulas with $|C_i| = K, \forall i$. Assume that each of the n variables occurs in at most $2^{\frac{k}{10}}$ clauses. Then $\exists$ a satisfying assignment for F.

PROOF:

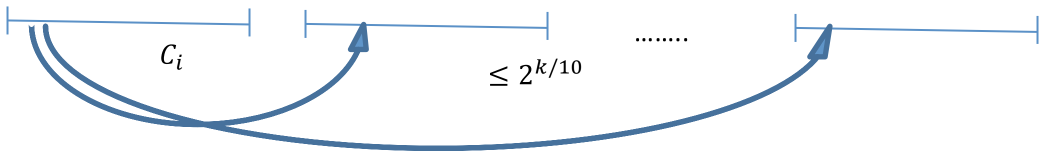
Give a random assignment to the variables.

Let E_i be the event that C_i is not satisfied.

$Prob(E_i) = 2^{-k}, \forall i.$

An event E_i might depend on another event E_j only if C_i and C_j share a common variable.

$\Longrightarrow$Any event might depend on at most $k2^{k/10}$ other events.



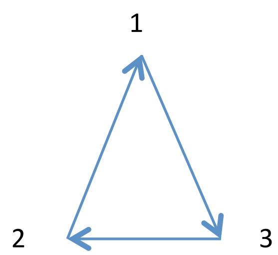
$$" ep(d+1) " = e \cdot 2^{-k}(k2^{k/10} + 1) \leq 1$$

$$\Longrightarrow Prob[\bigcap_{i=1}^{m} \overline{E_i}] > 0$$

$\Longrightarrow$ F is satisfiable. **End of Proof.**

EXAMPLE:

A tournament on n nodes is a complete graph $G(V, E)$. Each node is a player. $< i, j > \in E$ if player i has defeated player j. We say that the tournament has property P_k if for every subset of k players $\exists$ another player who has defeated all the k players.



Has property P_1

LEMMA:

For every k there is a finite tournament with property P_k.

PROOF:

Consider a tournament of size n where the edges have been generated randomly. Let X be a specific set of k players. Let y be another player.

$Prob[y\ has\ defeated\ all\ the\ players\ of\ X] = 2^{-k}$

$\Rightarrow Prob[y\ has\ not\ defeated\ X] = (1 - 2^{-k})$

$\Rightarrow Prob[none\ of\ the\ other\ players\ defeated\ X] \leq (1 - 2^{-k})^{n-k}$

$\Rightarrow Prob[\exists\ X\ with\ |X| = k\ ,$

$s.t. none\ of\ the\ other\ players\ defeated\ X] \leq \binom{n}{k}(1 - 2^{-k})^{n-k}$

If this $prob$ is < 1 then it means that $Prob[G\ satisfies\ P_k] > 0$.

What is the min. value of n for which $\binom{n}{k}(1 - 2^{-k})^{n-k} < 1$?

$(\frac{en}{k})^k e^{-(n-k)2^{-k}} < 1$

$ie\ (\frac{en}{k})^k < e^{n2^{-k}}$

Take $\log_e of\ both\ sides.$

$\Rightarrow \frac{n}{\log_e n} > k2^k$

$n = \Omega(k^2 2^k)$. **End of Proof.**