CSE6512 Lecture 10 Notes

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1. Searching in $O(1)$ time (M. Ajtai, J. Komlõs & E. Szemerêdi, 1985).

We continue our discussion on devising a data structure for a static input set such that searching for any element can be done in O(1) time. Two levels of hashing will be employed to store the input set.

Input: $S = \{ k_1, k_2, \ldots\ldots\ldots\ldots\ldots, k_s \} \subseteq M$.

Let $M = \{ 0, 1, \ldots\ldots\ldots\ldots\ldots, m-1 \}$ and $N = \{ 0, 1, \ldots\ldots\ldots\ldots\ldots, n-1 \}$.

W.L.O.G., Let $p = (m+1)$ be a prime number. $\mathbb{Z}_p = \{ 0, 1, \ldots\ldots\ldots\ldots\ldots, p-1 \}$.

For any $1 \le k \le m$, let $h_k(x) = kx \bmod p \bmod n$.

Let $V \subseteq M$ be any set where $|V| = v$.

Let $B_i(k, n, V)$ be the set of elements of V that are hashed into i by h_k, for $i \in N$.

$B_i(k, n, V) = \{ x \in V\colon\ h_k(x) = i,\ i = 0, 1, \ldots\ldots\ldots\ldots.., n-1 \}$.

Let $|B_i(k, n, V)| = b_i(k, n, V)$.

Lemma. $\sum_{k=1}^{m} \sum_{i=0}^{n-1} \binom{b_i(k,n,V)}{2} < \frac{mv^2}{n}$ for all $V \subseteq M$ and $n > v$.

Corollary. $\exists\, k, s.t. \sum_{i=0}^{n-1} \binom{b_i(k,n,V)}{2} < \frac{v^2}{n}$.

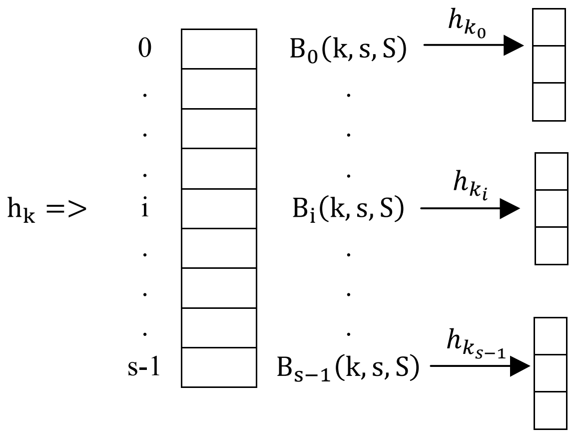
How do we store S?

There will be two levels of hashing. In the first level

use: $n = s, V = S$

Let h_k be a hash function that satisfies the following equation (1). The existence of such a function is ensured by the above Corollary. Assume that $\binom{a}{b} = 0$ when $a < b$.

$$\sum_{i=0}^{s-1} \binom{b_i(k,s,S)}{2} < \frac{s^2}{s} = s \qquad (1)$$



In the second level do the following:

- For the bucket i $(0 \le i \le s-1)$
  Use a hash function h_{k_i} with "n" = $b_i(k, s, S)^2$. In this case the hashing will be perfect (for an appropriate choice of h_{k_i}).

Space for the hash functions = $(s + 1)$.

Space for the first level = s.

Space for the second level = $\sum_{i=0}^{s-1} b_i(k, s, S)^2$.

From equation (1)

➔ $\sum_{i=0}^{s-1} [(b_i(k, s, S)]^2 < 2s + \sum_{i=0}^{s-1} (b_i(k, s, S) = 3s$.

➔ Total memory used = O(s).

Note.

Searching time = $O(1)$.

We only have to do two hash function evaluations.

How do we find good k values?

We have to find (s+1) hash functions such that for each function the above Corollary holds. If the set that is hashed is V with $|V| = v$, then we can try each value of k and this trivial algorithm takes $O(mn)$ time. This can be done in O(mv log v) time as well.

Fact.

For at least $\frac{1}{2}$ of the k-values $\sum_{\mathrm{i}=0}^{n-1}\binom{\mathrm{b_i}(k,n,V)}{2} < \frac{2v^2}{n}$.

- If we pick a random k, then Prob. $\left[\sum_{\mathrm{i}=0}^{n-1}\binom{\mathrm{b_i(k,n,V)}}{2} < \frac{2v^2}{n}\right] \geq \frac{1}{2}$.
- As a result, the time needed to find a good k is Õ $(n \log v)$.

Therefore, the time needed to find all the (s+1) hash functions is

$\tilde{O}([s + \sum_{i=0}^{s-1} b_i(k, s, S)^2] \log s) = \tilde{O}(s \log s)$.

The probabilistic method:

is used to show the existence of objects that possess a given set of properties.

We use two basic facts:

1. If X is a random variable with a mean μ then

   X takes on a value that is ≥ μ and X takes on a value that is ≤ μ.

2. Let U be a set of objects and let P be a property.

   If Prob. $[a\ random\ object\ of\ U\ has\ property\ P] > 0$ then

   it implies that U has at least one object with property P.

Example 1.

Let $G\ (V, E)$ be an undirected graph. Then

∃ a partition of V into V_1 and V_2, s.t. the number of edges from V_1 to V_2 is ≥ $\frac{|E|}{2}$.

Proof.

For every node $u \in V$

Put it in V_1 with probability = $\frac{1}{2}$;

Put it in V_2 with probability = $\frac{1}{2}$;

For any edge $e \in E$

Probability that it goes from V_1 to V_2 = $\frac{1}{2}$.

=> The expected number of edges from V_1 to V_2 is ≥ $\frac{|E|}{2}$.

Using (1), ∃ a partition for which the number of edges from V_1 to V_2 is ≥ $\frac{|E|}{2}$. □

Example 2.

Input: F= $C_1 \wedge C_2 \wedge C_3 \wedge \dots\dots\dots \wedge C_m$, which is a CNF Boolean formula on n variables.

Fact: There exists an assignment that satisfies $\geq \frac{m}{2}$ clauses.

Proof.

Let C_i be any clause with *k* literals. Give a random assignment to the variables.

Prob. [C_i is not satisfied] $= 2^{-k}$.

=> Prob. [C_i is satisfied] $= 1 - 2^{-k} \geq \frac{1}{2}$.

=> Expected number of satisfied clauses $= \frac{m}{2}$.

Using (1), ∃ an assignment that satisfies $\geq \frac{m}{2}$ clauses. □

Example 3.

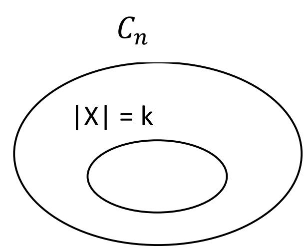
Let C_n be a complete graph on *n* nodes.

Let k and t be integers.

$R\ (k, t)$ is the minimum value of *n* s.t. if the edges of C_n are colored with red and blue, then for each such coloring ∃ either a red clique of size k or a blue clique of size t. *R*(*k, t*) is known as the Ramsey number.

Fact.

If $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$ then $R\ (k, k) > n$.



Proof.

Color the edges randomly.

Let X be a subset of nodes with $|X| = k$.

Prob. $[X\ is\ unicolored] = 2^{1-\binom{k}{2}}$.

=> Prob. [∃ a subset X of size k that is unicolored] $\leq \binom{n}{k} 2^{1-\binom{k}{2}}$.

If $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$ then

Prob. [No subset X of size k is unicolored] > 0.

=> ∃ A coloring under which no subset X of size k is unicolored.

=> $R\ (k, k) > \text{n}$. □

What is the maximum value of *n* for which $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$?

$\binom{a}{b} \approx (ae/b)^b$

$\left(\frac{ne}{k}\right)^k 2^{1-\binom{k}{2}} < 1$

$n^k = \frac{2^{k\frac{(k-1)}{2}}}{2} \left(\frac{k}{e}\right)^k$

$n^k \approx \left[2^{\frac{(k-1)}{2}} \frac{k}{e}\right]^k$

$n = 2^{\frac{(k-1)}{2}} \frac{k}{e}$ is a lower bound on $R(k, k)$. □